# FLOW OF NON-NEWTONIAN FLUID BETWEEN TWO COAXIAL CYLINDERS UNDER COMPLEX SHEAR CONDITIONS

S. A. Bostandzhiyan, V. I. Boyarchenko, and G. N. Kargopolova

UDC 532.542:532.135

The flow of a non-Newtonian fluid with an exponential rheological equation is investigated in the barrel of an extruder screw with consideration of the presence of circulating motion of the fluid in it.

In a hydrodynamic analysis of the flow of polymer melts in the barrel of an extruder screw, one usually uses a plane model of the screw, i.e., one investigates the flow of a non-Newtonian fluid with an exponential rheological equation between two parallel plates, one of which is stationary and the other moves at a constant velocity in a direction opposite to the movement of the plate, and a pressure gradient acts in the gap between the plates [1-3]. The plane model of a screw does not take into account the curvature of the screw barrel, whose effect on the characteristic of the screw and on the flow pattern increases with an increase of the degree of deviation of the properties of the material being processed from the properties of a Newtonian fluid and relative depth of the screw barrel. A cylindrical model of a screw was used for taking into account the curvature of the screw barrel in [4, 5]. This model represents two coaxial cylinders, one stationary and the other rotating with a constant angular velocity. A tangential pressure gradient acts angularly in a direction opposite to that of the rotation of the cylinder in the annular gap filled with fluid with an exponential rheological equation. Both these models assume that simple shear occurs during flow of the material. In the plane model the trajectories of the fluid particles are parallel straight lines, and in the cylindrical model the motion of the fluid particles occurs along concentric circles.

In reality fluid flow in the barrel of an extruder screw has a considerably more complex character. Since the tangential velocities of the fluid particles adjacent to the hub (in reversed motion) are directed at an angle to the axis of the screw barrel, then in addition to longitudinal flow along the axis of the screw barrel, which determines the output, there arises a circulating flow in a direction perpendicular to the axis of the screw channel. If the fluid is Newtonian, then by virtue of the linearity of the relations between the stress and strain rate tensors neither flow has an effect on the other and both flows can be treated separately. If the rheological equation is more complex, not linear, as in the power law, then generally speaking the transverse circulating flow will have an effect on the flow in a longitudinal direction and ultimately on the output. Complex shear was investigated in [6] in connection with the plane model of a screw. Here we will consider the flow of a non-Newtonian fluid with an exponential rheological equation between infinite coaxial cylinders under complex shear condition as related with the cylindrical model of a screw.

1. Let us consider the motion of a fluid with an exponential rheological equation in the barrel of an extruder screw (Fig. 1a). The velocity of the fluid particles can be decomposed into two components, one of which $v_x$ is directed along and the other $v_y$ across the barrel axis. We neglect the effect of the edges of the screw and for greater clarity we will henceforth consider reversed motion, considering that the casing of the extruder rotates about a stationary screw. To take into account the curvature of the screw barrel, we will use a cylindrical model of the extruder and investigate the fluid flow between two infinite coaxial cylinders (Fig. 1b). The inside cylinder of radius $R_1$ is stationary and the outer cylinder of radius $R_2$ rotates at a constant linear velocity $v_0$. The x and y axes correspond to the longitudinal and transverse

Institute of Chemical Physics, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 6, pp. 1069-1076, June, 1970. Original article submitted October 21, 1969.

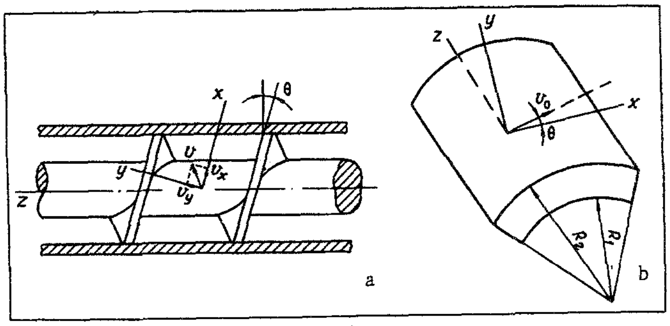



Fig. 1. Schematic diagram of the screw barrel and its cylindrical model.

directions relative to the axis of the screw channel. Let constant pressure gradients $\partial P/\partial x = A_1 > 0$ and $\partial P/\partial y = A_2 > 0$ act along these axes in the annular gap, whereby we will consider $A_1$ to be given and $A_2$ to be the unknown quantity.

The equations of motion in cylindrical coordinates have the form

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{r\varphi}) = \frac{\partial P}{\partial \varphi}, \quad \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = \frac{\partial P}{\partial z}. \tag{1}$$

Assuming that the pressure gradient along angle $\partial P/\partial\varphi$ acts on circles of radius $r_0 = (R_1 + R_2)/2$, we can write

$$\frac{\partial P}{\partial \varphi} = (A_1\cos\theta + A_2\sin\theta)\, r_0, \quad \frac{\partial P}{\partial z} = A_2\cos\theta - A_1\sin\theta. \tag{2}$$

Strictly speaking, the helix angle changes along the depth of the screw barrel. We will neglect the change of $\theta$ due to the radius. By $\theta$ we mean the value of the helix angle at an average radius.

Integrating (1) and taking into account (2), we obtain

$$\begin{gathered}\tau_{r\varphi} = \frac{A_1\cos\theta}{2}\, r_0\left(1+\frac{A_2}{A_1}\operatorname{tg}\theta\right)\left(1+\frac{R_1^2C_1}{r^2}\right), \\ \tau_{rz} = \frac{A_1\cos\theta}{2}\left(\frac{A_2}{A_1}-\operatorname{tg}\theta\right)\left(r+\frac{R_1^2C_2}{r}\right),\end{gathered} \tag{3}$$

where $C_1$ and $C_2$ are constants of integration.

The rheological equation in a general form is written so [1]:

$$\bar{\bar{\tau}} = \eta_{ef}\bar{\bar{\Delta}}. \tag{4}$$

The effective viscosity in the case being considered has the form

$$\eta_{ef} = \eta_0\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right]^2+\left(\frac{\partial v_z}{\partial r}\right)^2\right\}^{\frac{n-1}{2}}. \tag{5}$$

Taking into account (3), (4), and (5), we obtain the following system of equations:

$$\begin{gathered}\frac{A_1\cos\theta}{2}\, r_0\left(1+\frac{A_2}{A_1}\operatorname{tg}\theta\right)\left(1+\frac{R_1^2C_1}{r^2}\right) = \eta_0\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right]^2+\left(\frac{\partial v_z}{\partial r}\right)^2\right\}^{\frac{n-1}{2}} r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right), \\ \frac{A_1\cos\theta}{2}\left(\frac{A_2}{A_1}-\operatorname{tg}\theta\right)\left(r+\frac{R_1^2C_2}{r}\right) = \eta_0\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right]^2+\left(\frac{\partial v_z}{\partial r}\right)^2\right\}^{\frac{n-1}{2}}\frac{\partial v_z}{\partial r}.\end{gathered} \tag{6}$$

We introduce dimensionless quantities

$$\zeta = \frac{r}{R_1}, \quad b = \frac{R_2}{R_1}, \quad a = \frac{A_2}{A_1}, \quad \alpha = \frac{v_0}{R_1}\left(\frac{2\eta_0}{R_1A_1\cos\theta}\right)^{\frac{1}{n}},$$

$$v_{\mathrm{I}}=\frac{v_{\varphi}}{v_0},\quad v_{\mathrm{II}}=\frac{v_z}{v_0},\quad v_1=\frac{v_x}{v_0},\quad v_2=\frac{v_y}{v_0}.$$

System of Eqs. (6) can be written in a dimensionless form so:

$$\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\left(1+\frac{C_1}{\zeta^2}\right)=\alpha^n\left\{\left[\zeta\frac{\partial}{\partial\zeta}\left(\frac{v_{\mathrm{I}}}{\zeta}\right)\right]^2+\left(\frac{\partial v_{\mathrm{II}}}{\partial\zeta}\right)^2\right\}^{\frac{n-1}{2}}\zeta\frac{\partial}{\partial\zeta}\left(\frac{v_{\mathrm{I}}}{\zeta}\right),$$

$$(a-\mathrm{tg}\,\theta)\left(\zeta+\frac{C_2}{\zeta}\right)=\alpha^n\left\{\left[\zeta\frac{\partial}{\partial\zeta}\left(\frac{v_{\mathrm{I}}}{\zeta}\right)\right]^2+\left(\frac{\partial v_{\mathrm{II}}}{\partial\zeta}\right)^2\right\}^{\frac{n-1}{2}}\frac{\partial v_{\mathrm{II}}}{\partial\zeta}.$$

Dividing one equation by the other, we obtain the relation between the components of the strain rate tensor

$$(a-\mathrm{tg}\,\theta)\left(\zeta+\frac{C_2}{\zeta}\right)\zeta\frac{\partial}{\partial\zeta}\left(\frac{v_{\mathrm{I}}}{\zeta}\right)=\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\left(1+\frac{C_1}{\zeta^2}\right)\frac{\partial v_{\mathrm{II}}}{\partial\zeta}.$$

Using this relation, we can transform the system of equation to a form not containing nonlinear terms with velocity components:

$$\zeta\frac{\partial}{\partial\zeta}\left(\frac{v_{\mathrm{I}}}{\zeta}\right)=\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{f(\zeta,C_1,C_2,a)}{\alpha}\left(1+\frac{C_1}{\zeta^2}\right),$$
$$\frac{\partial v_{\mathrm{II}}}{\partial\zeta}=(a-\mathrm{tg}\,\theta)\frac{f(\zeta,C_1,C_2,a)}{\alpha}\left(\zeta+\frac{C_2}{\zeta}\right).\tag{7}$$

Here

$$f(\zeta,C_1,C_2,a)=\left[\left(\frac{1+b}{2}\right)^2(1+a\,\mathrm{tg}\,\theta)^2\left(1+\frac{C_1}{\zeta^2}\right)^2+(a-\mathrm{tg}\,\theta)^2\left(\zeta+\frac{C_2}{\zeta}\right)^2\right]^{\frac{1-n}{2n}}.$$

Thus, the system of equations was decomposed into two equations, which we can integrate separately.

Equations (7) must be solved with the following boundary conditions:

$$v_{\mathrm{I}}=v_{\mathrm{II}}=0\ \text{when}\ \zeta=1,\quad v_{\mathrm{I}}=1,\quad v_{\mathrm{II}}=0\ \text{when}\ \zeta=b.\tag{8}$$

We integrate Eqs. (7) and satisfy the first pair of boundary conditions (8). As a result of integration we obtain

$$v_{\mathrm{I}}=\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{\zeta}{\alpha}\int_1^{\zeta}f(\zeta,C_1,C_2,a)\left(1+\frac{C_1}{\zeta^2}\right)\frac{d\zeta}{\zeta},$$

$$v_{\mathrm{II}}=\frac{a-\mathrm{tg}\,\theta}{\alpha}\int_1^{\zeta}f(\zeta,C_1,C_2,a)\left(\zeta+\frac{C_2}{\zeta}\right)d\zeta.$$

The longitudinal and transverse velocity components in the screw barrel are expressed in terms of $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ by the formulas

$$v_1=v_{\mathrm{I}}\cos\theta-v_{\mathrm{II}}\sin\theta,\quad v_2=v_{\mathrm{I}}\sin\theta+v_{\mathrm{II}}\cos\theta.\tag{9}$$

We calculate the flow rate of the fluid in the direction of the axis of the screw barrel and in a transverse direction. Taking into account the assumption of the constancy of $\theta$, we can write the expressions for the flow rates in longitudinal and transverse directions per unit width of the barrel and length of the edge, respectively, so:

$$Q_1=\int_1^b v_1d\zeta,\quad Q_2=\int_1^b v_2d\zeta.$$

Replacing $v_1$ under the integral sign, according to (9) we obtain

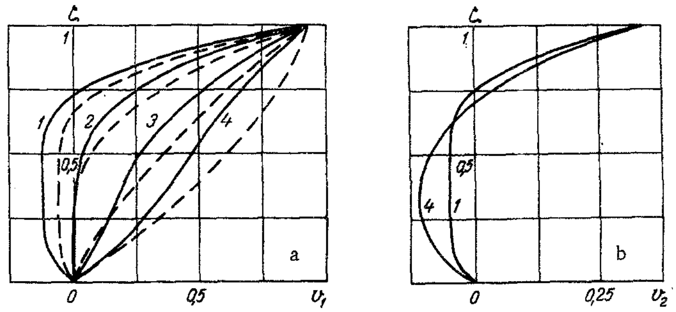



Fig. 2. Profiles at linear velocities $v_1$ (a) and $v_2$ (b) for different values of $\alpha$: 1) $\alpha = 0.7 \cdot 10^{-3}$; 2) $0.37 \cdot 10^{-2}$; 3) 0.101; 4) $\infty$.

$$Q_1 = \frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{\cos\theta}{\alpha}\int_1^b \zeta \int_1^\zeta f(\zeta, C_1, C_2, a)\left(1+\frac{C_1}{\zeta^2}\right)\frac{d\zeta}{\zeta}\,d\zeta$$
$$-(a-\mathrm{tg}\,\theta)\frac{\sin\theta}{\alpha}\int_1^b\int_1^\zeta f(\zeta, C_1, C_2, a)\left(\zeta+\frac{C_2}{\zeta}\right)d\zeta d\zeta.$$

After changing the order of integration we have

$$Q_1 = \frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{\cos\theta}{2\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(1+\frac{C_1}{\zeta^2}\right)(b^2-\zeta^2)\,d\zeta$$
$$-(a-\mathrm{tg}\,\theta)\frac{\sin\theta}{\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(\zeta+\frac{C_2}{\zeta}\right)(b-\zeta)\,d\zeta.$$

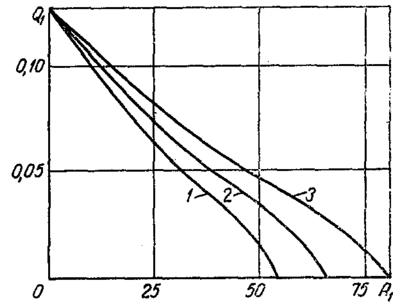
Proceeding analogously with the second integral, for flow rate $Q_2$ we obtain

$$Q_2 = \frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{\sin\theta}{2\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(1+\frac{C_1}{\zeta^2}\right)(b^2-\zeta^2)\,d\zeta$$
$$+(a-\mathrm{tg}\,\theta)\frac{\cos\theta}{\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(\zeta+\frac{C_2}{\zeta}\right)(b-\zeta)\,d\zeta.$$

The constants of integration $C_1$ and $C_2$ and parameter $a$ remain unknown. They are found as a result of satisfying the second pair of boundary conditions (8) and from the condition of equating the transverse flow rate of the fluid to zero. Thus, to find $C_1$, $C_2$, and $a$ we arrive at the following system of equations:

$$\begin{gathered}
\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{b}{\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(1+\frac{C_1}{\zeta^2}\right)\frac{d\zeta}{\zeta} = 1,\\
\int_1^b f(\zeta, C_1, C_2, a)\left(\zeta+\frac{C_2}{\zeta}\right)d\zeta = 0,\\
\frac{1+b}{2}(1+a\,\mathrm{tg}\,\theta)\frac{\sin\theta}{2\alpha}\int_1^b f(\zeta, C_1, C_2, \alpha)\left(1+\frac{C_1}{\zeta^2}\right)(b^2-\zeta^2)\,d\zeta\\
+(a-\mathrm{tg}\,\theta)\frac{\cos\theta}{\alpha}\int_1^b f(\zeta, C_1, C_2, a)\left(\zeta+\frac{C_2}{\zeta}\right)(b-\zeta)\,d\zeta = 0.
\end{gathered} \tag{10}$$

2. The values of $C_1$, $C_2$, and $a$ were found and the velocities $v_1$, $v_2$, and volume flow rate $Q_1$ were calculated on a computer by numerical methods. The results were compared with those obtained earlier in [4], where fluid motion between two cylinders under simple shear conditions was considered. In this



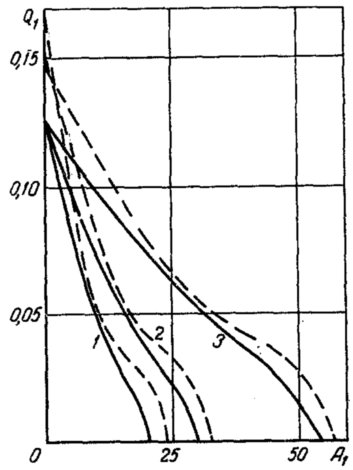




Fig. 3 Fig. 4

Fig. 3. Curves of the dependence of the dimensionless flow rate $Q_1$ on the pressure gradient $A_1$ (G/cm$^3$) for different velocities $v_0$: 1) $v_0 = 12.5$ cm/sec; 2) 25; 3) 50.

Fig. 4. Dependence of the dimensionless flow rates $Q_1$ on the pressure gradient $A_1$ (G/cm$^3$) in the case of simple and complex shears for different values of n: 1) n = 0.81; 2) 0.22; 3) 0.29.

study the rheological equation of the fluid was written in the form

$$r\frac{d\omega}{dr} = k|\tau|^{m-1}\tau.$$

To establish a correspondence between the cases of simple and complex shear we must set

$$n = \frac{1}{m}, \quad \eta_0 = k^{-\frac{1}{m}}, \quad v_0 = \frac{\omega_0 R_2}{\cos\theta},$$

where $\omega_0$ is the angular velocity of rotation of the outer cylinder in the case of simple shear. The calculations were performed for the following values of the parameters: $\theta = 20°$, $R_1 = 2$ cm, $R_2 = 2.5$ cm, $\omega_0 = 4.7$ sec$^{-1}$, m = 3.5, k = 0.01 (cm$^2$/G) sec$^{-1}$.

In Fig. 2a the profiles of the velocity components along the axis of the screw channel are shown by the solid lines and the corresponding profiles in the case of simple shear are shown by dashed lines for different values of the dimensionless parameter $\alpha$. The difference between the corresponding curves increases with increase of $\alpha$. In the region of the values of parameters at which the gradient of the longitudinal velocity does not change sign inside the gap, the profiles of these velocities have a point of inflection in the case of complex shear. The phenomenon of the inflection is explained by the effect of the transverse velocity on the longitudinal by way of the effective viscosity. Figure 2b shows the transverse velocity profiles. As we see from the figure, the dependence of $v_2$ on $\alpha$ is weak.

Figure 3 shows the effect of the velocity of rotation of the outer cylinder on the dependence of the dimensionless flow rate $Q_1$ along the x axis on the pressure gradient $A_1$. The point $Q_1 = 0$ corresponds completely to closed emergence from the extruder. When $A_1 = 0$ the dimensionless flow rate does not depend on the rotational velocity.

For comparison of the results obtained in simple and complex shears, Fig. 4 shows the curves of the dependence of the dimensionless flow rate of the fluid $Q_1$ on the pressure gradient $A_1$ for different values of n. The solid lines correspond to complex shear and the dashed lines to simple. We see from the figure that for pseudoplastic fluids at the same values of the pressure gradient, the flow rate in the case of complex shear is less than in the case of simple shear. This is explained by the fact that during flow of a fluid under complex shear conditions its effective viscosity is less than under simple shear conditions, which leads to an increase of the effect of counterpressure. For dilitant fluids the effective viscosity and the flow rate in the case of complex shear will be greater than the corresponding values in the case of simple shear.

A characteristic feature of flow under complex shear conditions is the weaker dependence of the dimensionless flow rate on n in the absence of counterpressure (in Fig. 4 the solid curves originate from different points on the y axis, but owing to the smallness of the scale the latter merge). On the whole the character of the dependence of the flow rate on the pressure gradient in complex shear is more monotonic than in simple shear.

Thus, the flow of an exponential fluid between two cylinders under complex shear conditions gives a different picture of the velocity distribution than in the case of simple shear, which at certain values of the parameters can have an effect on the flow rate. Consideration of the transverse flow is necessary if one is considering the problem of heat transfer in the barrel of an extruder screw, since the contribution of the transverse component of velocity to dissipative heating is considerable.

## NOTATION

| | |
|---|---|
| $r, \varphi, z$ | are the cylindrical coordinates; |
| $x, y$ | are the directions along and across the screw barrel; |
| $R_1, R_2$ | are the inner and outer radii of the cylinders; |
| $r_0$ | is the average radius; |
| $\Theta$ | is the helix angle; |
| $\underline{\underline{\tau}}$ | is the stress tensor; |
| $\underline{\underline{\Delta}}$ | is the strain rate tensor; |
| $\tau_{r\varphi}, \tau_{zz}$ | are the components of the stress tensor; |
| $\eta_{ef}$ | is the effective viscosity; |
| $\eta_0, n$ | are the rheological constants; |
| $v_0$ | is the linear velocity of points of the outer cylinder; |
| $P$ | is the pressure; |
| $A_1, A_2$ | are the pressure gradients; |
| $\xi$ | is the dimensionless radius; |
| $v_I, v_{II}, v_1, v_2$ | are the dimensionless velocity components along axes $\varphi$, z, x, and y; |
| $b$ | is the radii ratio; |
| $a$ | is the pressure gradient ratio; |
| $\alpha$ | is the dimensionless parameter; |
| $Q_1, Q_2$ | are the dimensionless volume flow rates along and across axis of barrel. |

## LITERATURE CITED


1. J. M. McKelvey, Plastic Processing, John Wiley and Sons, New York (1962).
2. S. A. Bostandzhiyan and A. M. Stolin, Izv. Akad. Nauk SSSR, Mekhanika, No. 1, 185 (1965).
3. R. V. Torner, L. F. Gudkova, and I. K. Nikolaev, Mekh. Polim., No. 6, 138 (1965).
4. S. A. Bostandzhiyan and V. I. Boyarchenko, Mekh. Polim., No. 6, 1094 (1968).
5. Z. Tadmor, Polim. Engin. and Sci., July, 203 (1966).
6. S. A. Bostandzhiyan, V. I. Boyarchenko, and G. N. Kargopolova, in: Rheophysics and Rheodynamics of Flowing Systems [in Russian], Nauka i Tekhnika, Minsk (1970).